Absolute Value and Reciprocal Functions Unit Exam Solutions

Use the following information to answer the first question.

Given the following table of values for y = f(x)

x	y
-2	-19
-1	-15
0	-11
1	-7
2	-3

Consider the possible table of values for y = $|f(x)|$

A			B			C			D	
x	y		x	y		x	y		x	y
-2	19		2	19		-2	-19		2	-19
-1	15		1	15		-1	-15		1	-15
0	11		0	11		0	-11		0	-11
1	7		1	7		-1	-7		1	-7
2	3		2	3		-2	-3		2	-3

1. The correct table of values for y = |f(x)| is

A) A B) B C) C D) D

Solution

When determining an absolute value function, it is important to remember that for a given value of x in the original function, it is the absolute value of the y coordinate that is taken.

In table A, all of the x-coordinates are the same, and their corresponding y-coordinates have all changed from negative to positive. This means that the absolute value of all these y-coordinates has been taken. The correct table is A.

The correct answer is A.

Use the following information to answer the next question.

Consider the following statements.

Statement 1	The following 5 numbers are ordered from least to greatest: $\lvert 0.7\rvert$, 0.9, $\lvert -1.5\rvert$, 3.1, $\lvert\frac{-11}{2}\rvert$
Statement 2	The value of $\lvert -6 - 2(4)\rvert$ is 2.
Statement 3	$3\lvert 2 - 5\rvert + -4\lvert 1 - (-2)\rvert = -3$.
Statement 4	The y-intercept of $y = \lvert 3x - 12\rvert$ is -12.

2. The two true statements are

A) 1 and 2 B) 3 and 4 C) 1 and 3 D) 2 and 4

Solution

<u>Statement 1</u>

Express all 5 numbers as their rational number equivalents.

$\lvert 0.7\rvert = 0.7$

$0.9 = 0.9$

$\lvert -1.5\rvert = 1.5$

$3.1 = 3.1$

$\lvert -5.5\rvert = 5.5$

These 5 numbers are in order from least to greatest. Statement 1 is true.

<u>Statement 2</u>

$\lvert -6 - 2(4)\rvert$

Apply order of operations inside the absolute value symbols.

$\lvert -6 - 8\rvert$

$\lvert -14\rvert$

|-14| = 14

Statement 2 is false.

Statement 3

3|2 - 5| + -4|1 – (-2)|

Determine the value inside the absolute value symbols, and then multiply by the number in front of these symbols.

3|-3| + -4|3|

3|3| + -4|3|

9 + -12

= -3

Statement 3 is true.

Statement 4

To determine the y-intercepts, set x = 0 and solve for y.

y = |3x - 12|

y = |3(0) - 12|

y = |-12|

y = 12

The y-intercept is 12.

Statement 4 is false.

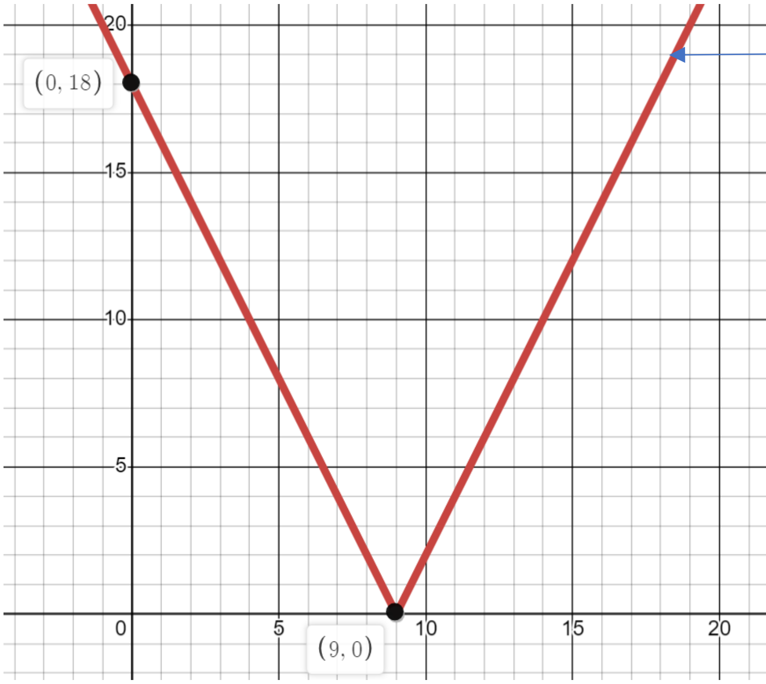
The correct answer is C.

3. The absolute value equation, y = |2x - 18| expressed as a piecewise function is

$$y = 2x - 18, \text{ if } x \geq K$$
$$y = -(2x - 18), \text{ if } x < K.$$

The value of K is _9_.

Solution



If this line continued down below the x-axis, the equation would be y = 2x – 18. At the x-intercept (9,0), the graph rises to the left because the x values less than 9 that originally generated negative y values are now generating positive y values because of the absolute value.

The key point is the x-intercept.

Therefore, when x is greater than or equal to 9, the equation is y = 2x – 18. When x is less than 9, the equation is y = -(2x – 18).

Expressed as a piecewise function,

$y = 2x - 18$, if $x \geq 9$, and

$y = -(2x - 18)$, if $x < 9$.

The value of K is 9.

4. Which of the following equations has no solution?

   A) $|-x + 8| - 2 = -1$
   B) $|\frac{1}{2}x - 12| + 5 = 7$
   C) $|4x + 1| - 10 = 0$
   D) $|-3x - 3| + 6 = 1$

Solution

If an absolute value expression is set equal to a negative number, there will be no solution. The reason for this is that an absolute value expression has to be positive by definition.

The equivalent equations are:

|-x + 8| = 1

$|\frac{1}{2}x - 12| = 2$

|4x + 1| = 10

|-3x - 3| = -5

The only absolute value expression not set equal to a positive number is the last one.

The correct answer is D.

5. The extraneous root for the equation |x + 1| = 2x – 2 is

A) -3 B) 3 C) $-\frac{1}{3}$ D) $\frac{1}{3}$

Solution

There are two equations to solve. Take the positive and negative quantity inside the absolute value symbols.

<u>x + 1 = 2x – 2</u>	<u>-(x+1) = 2x – 2</u>
1 = x – 2	-x – 1 = 2x – 2
3 = x	1 = 3x
	$\frac{1}{3} = x$

Verify these two solutions.

x = 3	x = $\frac{1}{3}$
|x + 1| = 2x – 2	|x + 1| = 2x – 2
| (3) + 1| = 2(3) – 2	$\|(\frac{1}{3}) + 1\| = 2(\frac{1}{3}) - 2$
|4| = 4	$\|\frac{4}{3}\| = \frac{2}{3} - \frac{6}{3}$
4 = 4	$\frac{4}{3} \neq -\frac{4}{3}$

The extraneous root is 1/3.

The correct answer is D

6. The solution(s) to $|x - 7| = x^2 - x - 42$ is/are

A) 7   B) 7, -7   C) 7, -7, -5   D) 7, -7, -5, 5

Solution

There are two equations to solve. Take the positive and negative quantity inside the absolute value symbols.

$\underline{x - 7 = x^2 - x - 42}$

$0 = x^2 - 2x - 35$

$0 = (x - 7)(x + 5)$

$x = 7$ and -5

$\underline{-(x - 7) = x^2 - x - 42}$

$-x + 7 = x^2 - x - 42$

$0 = x^2 - 49$

$0 = (x + 7)(x - 7)$

$x = -7$ and 7

Verify

$\underline{x = 7}$

$|x - 7| = x^2 - x - 42$

$|(7) - 7| = (7)^2 - (7) - 42$

$|0| = 49 - 7 - 42$

$0 = 0$

$\underline{x = -5}$

$|x - 7| = x^2 - x - 42$

$|(-5) - 7| = (-5)^2 - (-5) - 42$

$|-12| = 25 + 5 - 42$

$12 \neq -12$

$\underline{x = -7}$

$|x - 7| = x^2 - x - 42$

$|(-7) - 7| = (-7)^2 - (-7) - 42$

$|-14| = 49 + 7 - 42$

$14 = 14$

The solutions are 7 and -7.

The correct answer is B.

7. A school is running a contest to guess the number of round hard candies that are in a large jar. If the exact number happens to be 316 and a potential winning guess must be within ± 4, which absolute value equation will model this situation? [Let G = Guess]

   A) |G - 4| ≤ 316
   B) |G + 4| ≤ 316
   C) |G - 316| ≤ 4
   D) |G + 316| ≤ 4

Solution

A potential winning guess would be a number between 320 (316 + 4) and 312 (316 – 4). The numbers 312, 313, 314, 315, 316, 317, 318, 319, and 320, will satisfy the equation

|G - 316| ≤ 4.

The correct answer is C.

Use the following information to answer the next question.

If f(x) = 3x – 8, then consider the following statements regarding y = $\frac{1}{f(x)}$

Statement 1	The equation of the vertical asymptote is $x = \frac{8}{3}$.
Statement 2	The invariant points are (3,1) and (7, -1).
Statement 3	The y-intercept is (0, 0.125).
Statement 4	There are no x-intercepts.

8. The two true statements are

A) 1 and 2 B) 3 and 4 C) 2 and 3 D) 1 and 4

Solution

<u>Statement 1</u>

The reciprocal of f(x) is y = $\frac{1}{3x-8}$. To find the equation of the vertical asymptote, set the denominator equal to zero and solve for x.

3x – 8 = 0

3x = 8

x = $\frac{8}{3}$

Statement 1 is true.

Statement 2

For reciprocal functions, invariant points (points that do not change when a graph is transformed) occur when y = 1 or y = -1. The reason is that when 1 is reciprocated, it is still 1. When -1 is reciprocated, it is still -1.

To find invariant points, set substitute y = 1 and y = -1 into y = f(x).

(1) = 3x – 8

9 = 3x

3 = x

One invariant point is (3, 1).

(-1) = 3x – 8

7 = 3x

$\frac{7}{3} = x$

The other invariant point is $\left(\frac{7}{3}, -1\right)$.

Statement 2 is false.

Statement 3

Given y = $\frac{1}{3x-8}$, to find the y-intercept, set x = 0 and solve for y.

y = $\frac{1}{3(0)-8}$

y = $\frac{1}{-8}$.

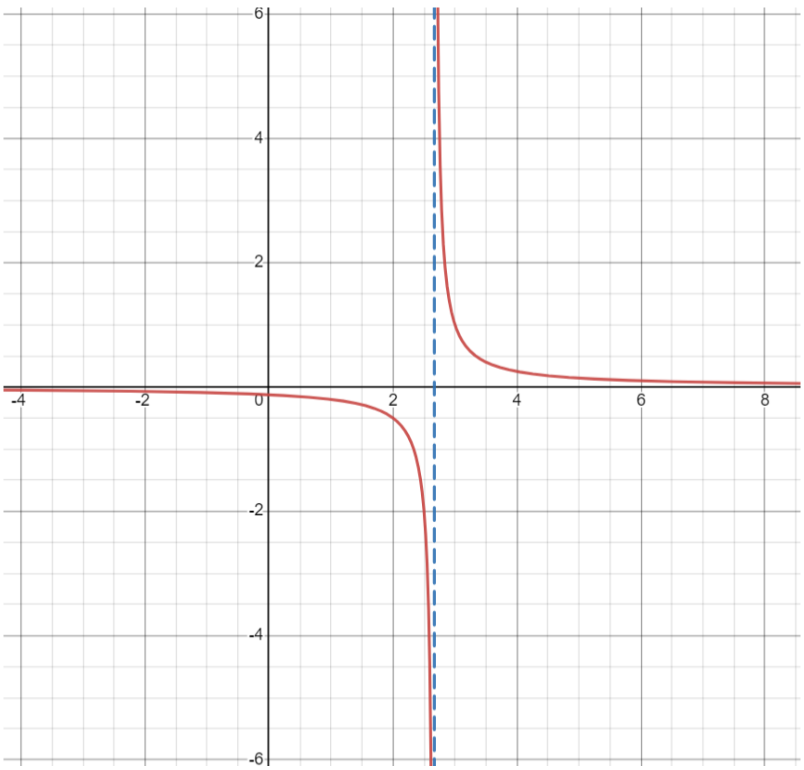
y = -0.125

Statement 3 is false.

Statement 4

The graph of this reciprocal function shown below indicates a horizontal asymptote of

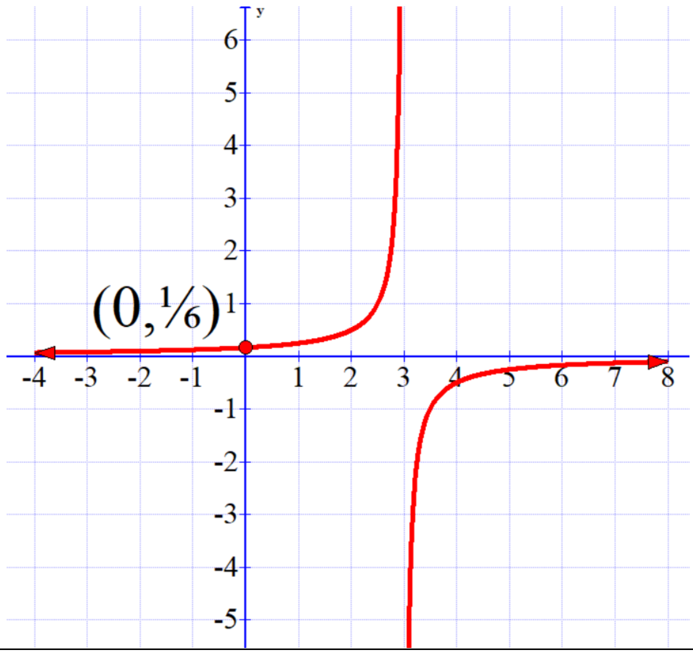
y = 0. This means that there is no value for x that would produce a y value of 0. Hence, there are no x-intercepts. This statement is true.



The correct answer is D.

Use the graph below to answer the next question.

The graph is the reciprocal function, y = $\frac{1}{f(x)}$. The equation of the vertical asymptote is x = 3.



9. The equation of y = f(x) is

A) y = 2x + 6 B) y = -2x + 6 C) y = x – 3 D) y = x + 3

Solution

Since the point $\left(0, \frac{1}{6}\right)$ is on the reciprocal function, the point (0,6) is on the original function. With an equation of the vertical asymptote being x = 0, the point (3,0) is on the original function.

The slope between these two points is -2 and the y-intercept is 6. The equation of y = f(x) in slope-y-intercept form is y = -2x + 6.

The correct answer is B.

Use the following information to answer the next question.

Analyze the vertical asymptotes for the following reciprocal functions.

I	II	III	IV
$f(x) = \frac{1}{6x - 12}$	$f(x) = \frac{1}{x^2 - x - 6}$	$f(x) = \frac{1}{(x-2)(x+7)}$	$f(x) = \frac{1}{x - 2}$

10. The function **not** having a vertical asymptote of x = 2 is

A) I B) II C) III D) IV

Solution

Factor denominators.

f(x) = $\frac{1}{6(x-2)}$

f(x) = $\frac{1}{(x-3)(x+2)}$

f(x) = $\frac{1}{(x-2)(x+7)}$

f(x) = $\frac{1}{(x-2)}$

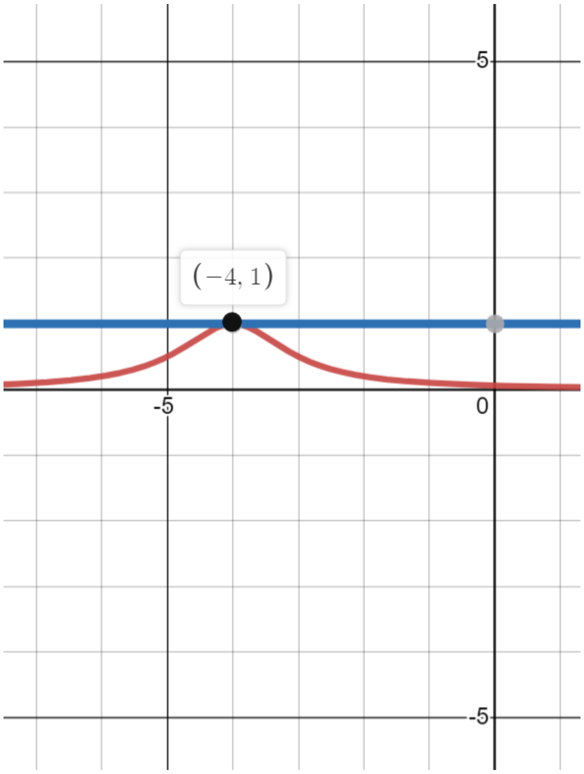
A binomial factor of (x – 2) indicates a vertical asymptote of x = 2. The only function not having this binomial factor is II.

The correct answer is B.

11. Given f(x) = x^2 + 8x + 17, and it's reciprocal function $y = \frac{1}{f(x)}$, there will be one invariant point in quadrant two (-x,y). The value of x is _4_.

Solution

The graph below shows $y = \frac{1}{x^2+8x+17}$ and the graph of y = 1. Invariant points occur when y = 1 and y = -1. Since the reciprocal graph does not go below the x-axis, there are no invariant points for y = -1. The intersection of the two graphs is the invariant point, which is (-4,1).



An alternative method is to set y = 1 for the original function and solve for x.

$y = x^2 + 8x + 17$

$(1) = x^2 + 8x + 17$

$0 = x^2 + 8x + 16$

$0 = (x + 4)^2$

$x = -4$

In the form (-x,y), the value of x is 4.

12. If f(x) = x^2 – 25 and g(x) = x^2 – 17x + 60, then y = $\frac{1}{f(x)}$ and y = $\frac{1}{g(x)}$ have one common non-permissible value, which is _5_.

Solution

Factor.

$$y = \frac{1}{f(x)} = \frac{1}{(x+5)(x-5)}$$

$$y = \frac{1}{g(x)} = \frac{1}{(x-5)(x-12)}$$

The common binomial factor is (x – 5). This means that the non-permissible value is 5.

13. If the point (4, $\frac{1}{5}$) is on y = f(x), then the corresponding point on $y = \frac{1}{f(x)}$ is

A) $(\frac{1}{4}, \frac{1}{5})$ B) $(\frac{1}{4}, 5)$ C) (4, 5) D) $(4, -\frac{1}{5})$

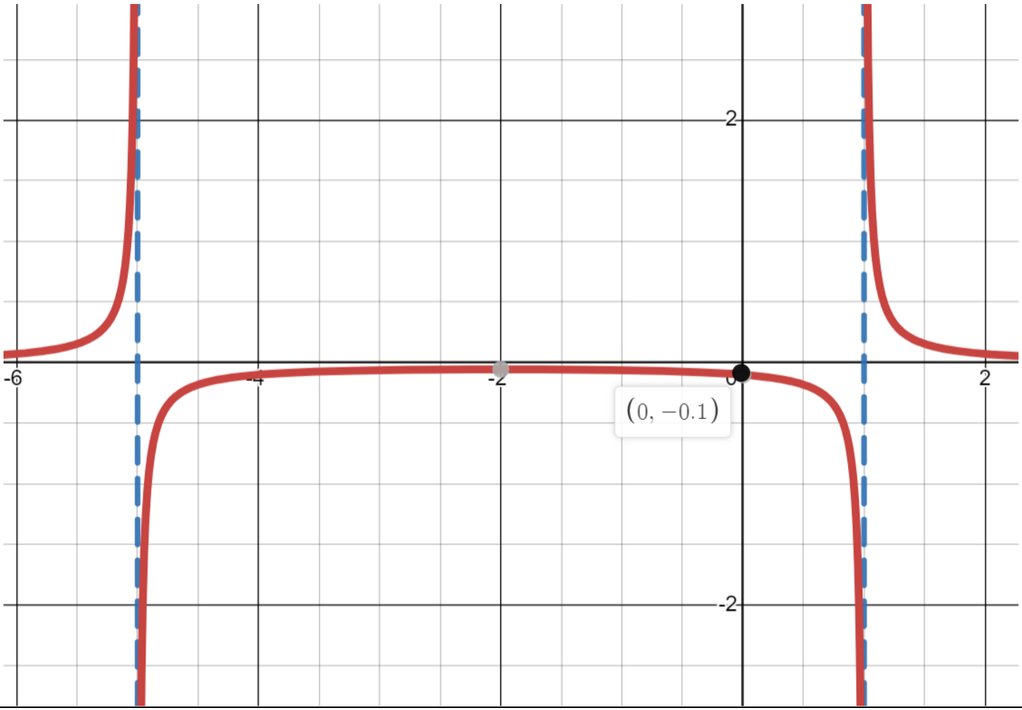
Solution

For a given point (x,y) on an original function, y = f(x), the corresponding point on the reciprocal function is (x, $\frac{1}{y}$). In other words, the value for x does not change. It is the value for y that is reciprocated. The corresponding point in this question is (4,5).

The correct answer is C.

Use the following graph to answer the next question.

Consider the graph of $y = \frac{1}{f(x)}$ shown below. The y-intercept is (0, -0.1) and the equations of the vertical asymptotes are x = -5 and x = 1.



14. When y = f(x) is written in the form, y = a (x – b) (x + c), the value for *a* is _2_.

Solution

The equations of the vertical asymptotes will help us determine the values of b and c. The binomial (x – b) indicates an asymptote to the right of the origin (0,0). Since the equation of the asymptote is x = 1, the corresponding binomial is (x – 1). Thus, be = 1.

The binomial (x + c) indicates an asymptote to the left of the origin. Since the equation of the asymptote on that side is x = -5, the corresponding binomial is (x + 5).

The point given on the reciprocal function of the graph is (0, -0.1). When the y value is reciprocated (to return to the original value of on y = f(x)), the point is (0, -10).

Substitute these 3 values into the equation to find a.

y = a (x – b) (x + c)

(-10) = a ((0) – (1)) ((0) + (5))

-10 = a (-1) (5)

$-10 = -5a$

$2 = a$

The value for a is 2.

Written Response

- Write your responses as neatly as possible.
- For full marks, your responses must address **all** aspects of the question.
- All responses, including descriptions and/or explanations of concepts must include pertinent ideas, calculations, formulas, and correct units.
- Your responses must be presented in a in a well-organized manner. For example, you may organize your responses in point form or paragraphs.

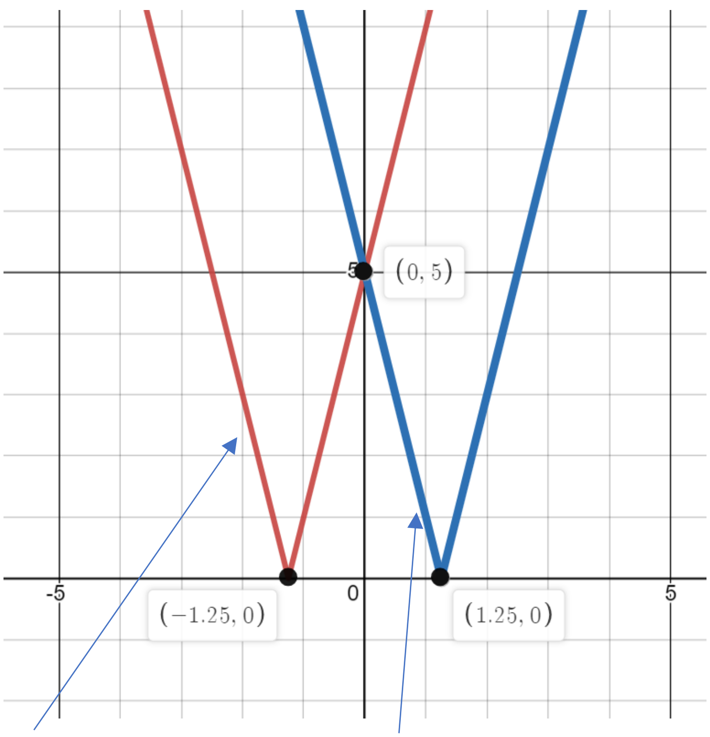
WRITTEN RESPONSE 1

- **Illustrate** how the absolute functions, f(x) = |4x + 5| and g(x) = |4x - 5| **compare** in terms of intercepts, domain and range. [2 Marks]

Illustrate: “Make clear by giving an example. The form of the example will be specified in the question: e.g., a word description, sketch, or diagram”.

Compare: “Examine the character or qualities of two things by providing characteristics of both that point out their mutual similarities and differences”.

Possible Solution



f(x) = |4x + 5| g(x) = |4x - 5|

The x-intercept for y = f(x) is -1.25 and the x-intercept for y = g(x) is 1.25.

The have the same y-intercept at 5.

They have the same domain.

$$[-\infty, \infty]$$

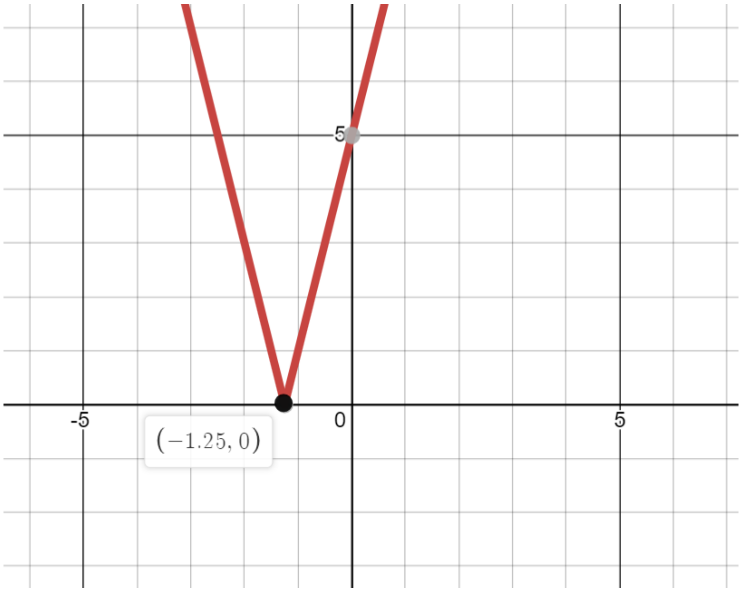
They have the same range.

$$[0, \infty]$$

- Express f(x) = |4x + 5| as a piecewise function. **Explain.** [2 Marks]

Explain: "Make clear what is not immediately obvious or entirely known; give the cause of or reason for; make known in detail".

Possible Solution



For all values of x greater than or equal to -1.25, the y values are positive, and all points lie on the line y = 4x + 5. The slope is 4 and the y-intercept is 5 (read from the values in the slope-y-intercept form of a linear equation).

For all values of x less than -1.25, the y values that were originally negative on the equation y = 4x + 5, have no become positive due to taking their absolute value.

Therefore, as a piecewise function, f(x) = |4x+5| is:

y = 4x + 5, if x ≥ -1.25

y = -(4x + 5), if x < -1.25.

- **Interpret** |4x - 5| < 0, in terms of a solution. [1 Mark].

Interpret: "Provide a meaning of something; present information in a new form that adds meaning to the original data".

Possible Solution

Any absolute value equation of the form |f(x)| = a, where a < 0, has no solution since by definition |f(x)| ≥ 0.

- **Solve** the absolute value equation, |4x + 5| = 9, **algebraically** and using technology (include a **sketch**). **Verify**. [3 Marks]

Solve: "Give a solution to a problem".

Algebraically: "Using mathematical procedures that involve variables or symbols to represent values".

Sketch: "Provide a drawing that represents the key features or characteristics of an object or graph".

Verify: "Establish, by substitution for a particular case or by geometric comparison, the truth of a statement".

Possible Solution

To solve algebraically, consider the two cases.

Either 4x + 5 = 9 or -(4x + 5) = 9

Case 1

4x + 5 = 9

4x = 4

x = 1

Case 2

-4x – 5 = 9

-4x = 14

$x = -\frac{14}{4} or -\frac{7}{2} or -3.5$

Verify x = 1

|4x + 5| = 9

|4(1) + 5| = 9

|9| = 9

9 = 9

Verify x = -3.5

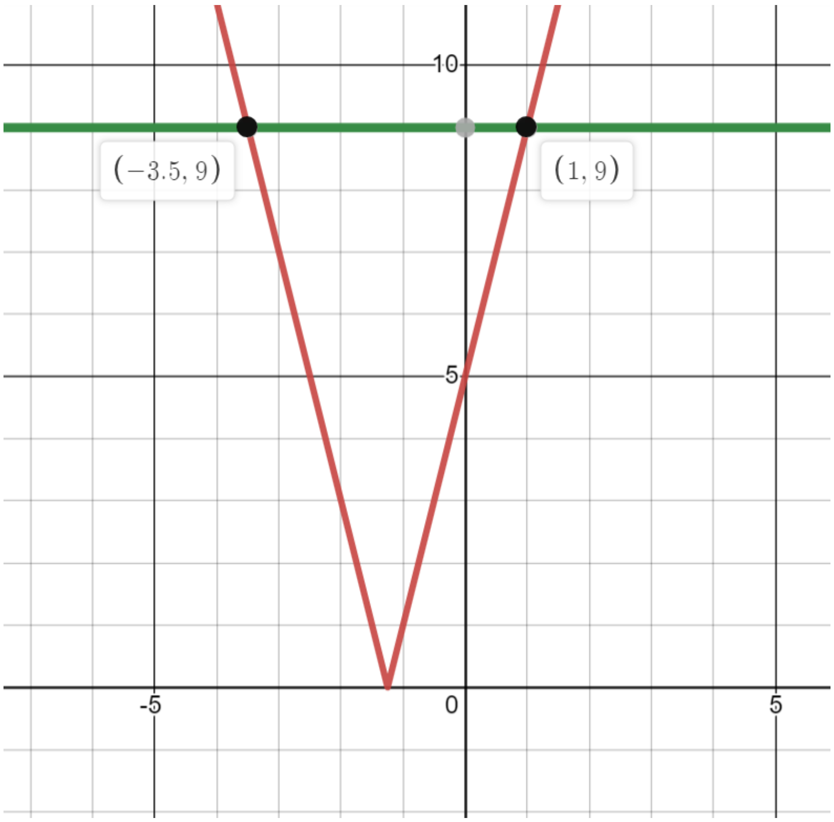
|4x + 5| = 9

|4(-3.5) + 5| = 9

|-9| = 9

9 = 9

To solve graphically, graph $y_1 = |4x + 5|$ and $y_2 = 9$ and find the x-coordinate(s) of the intersection point(s).

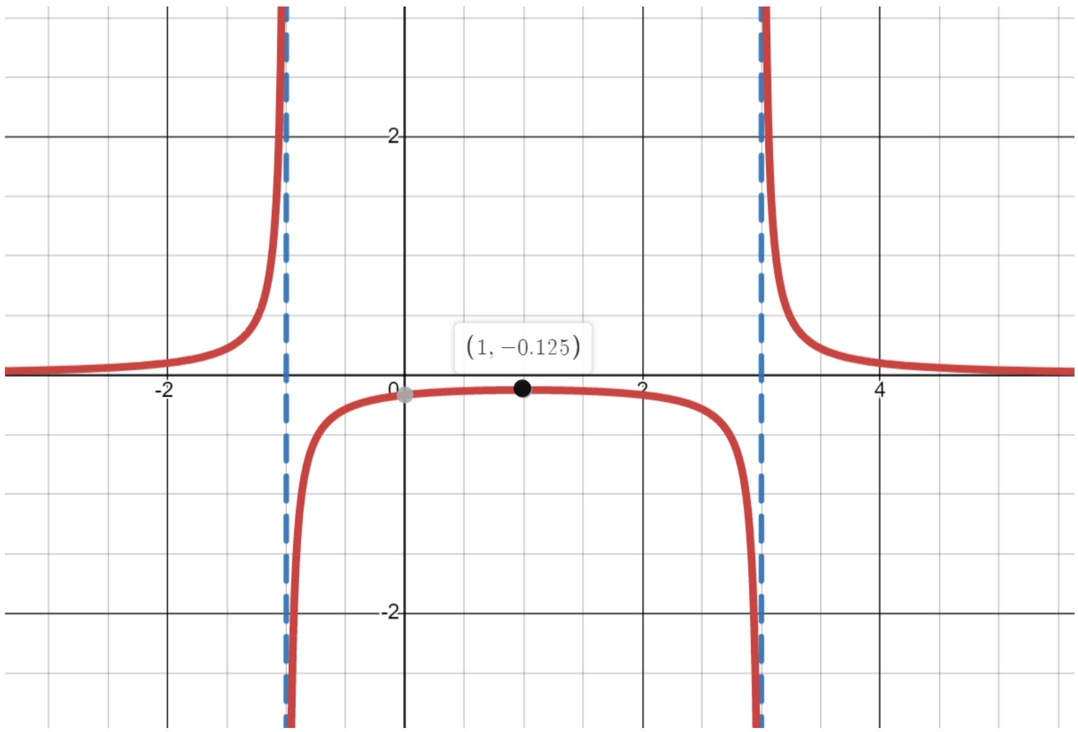


The solutions are x = 1 and x = -3.5.

WRITTEN RESPONSE 2

Use the following to answer the next question.

A math student was given the following graph of a reciprocal function $y = \frac{1}{f(x)}$ and asked to find the equation of y = f(x).



The student's work is shown below.

Step 1	$y = \frac{1}{a(x+3)(x+1)}$
Step 2	$-0.125 = \frac{1}{a((1)+3)((1)+1)}$
Step 3	$-0.125 = \frac{1}{a(4)(2)}$
Step 4	$8a = \frac{1}{-0.125}$
Step 5	a = -1
Step 6	f(x) = -(x + 3)(x + 1)

- **Analyze** the math student's work. **Determine** and correct the error. [2 Marks]

Analyze: "Make a mathematical examination of parts to determine the nature, proportion, function, interrelationships, and characteristics of the whole".

Determine: "Find a solution, to a specified degree of accuracy, to a problem by showing appropriate formulas, procedures, and/or calculations".

Possible Solution

There is a problem with step one. Since the equations for the vertical asymptotes are

x = 3 and x = -1, the corresponding binomial factors should be (x – 3) and (x + 1). The following steps to find the value of 'a' by substituting the point on the graph (1, -0.125) is the correct procedure.

$$y = \frac{1}{a(x-3)(x+1)}$$

$$-0.125 = \frac{1}{a((1)-3)((1)+1)}$$

$$-0.125 = \frac{1}{a(-2)(2)}$$

$$-0.125 = \frac{1}{-4a}$$

0.5a = 1

a = 2

The equation of y = f(x) is f(x) = 2(x – 3) (x + 1).

- State the equations of the vertical asymptotes and **describe** how they relate to the non-permissible values. [1 Mark]

Describe: "Give a written account of a concept".

Possible Solutions

The equations of the vertical asymptotes are x = 3 and x = -1. When the binomial equivalent for the vertical asymptote is (x – 3), the value making the denominator equal to zero is 3. This value is a non-permissible value because division by zero is undefined.

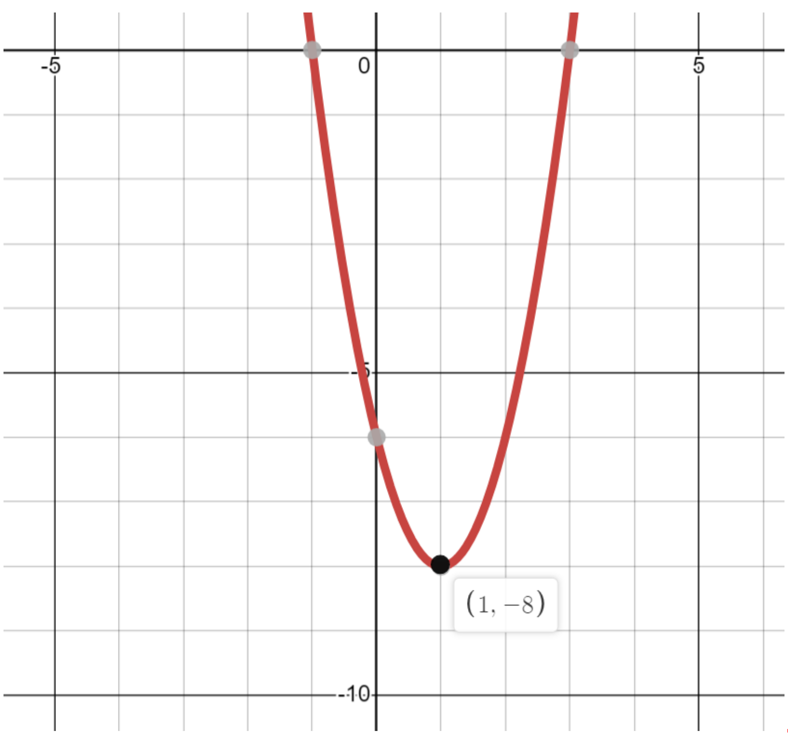
When the binomial equivalent for the vertical asymptote is (x + 1), the value making the denominator equal to zero is -1. This value is a non-permissible value because division by zero is undefined.

- **Compare** the ranges of y = f(x) and y = $\frac{1}{f(x)}$. [1 Mark]

Possible Solution

The equation of f(x) is y = 2(x – 3) (x + 1)

The graph of f(x) is shown below.



The range of the graph of y = f(x) is $[-8, \infty]$

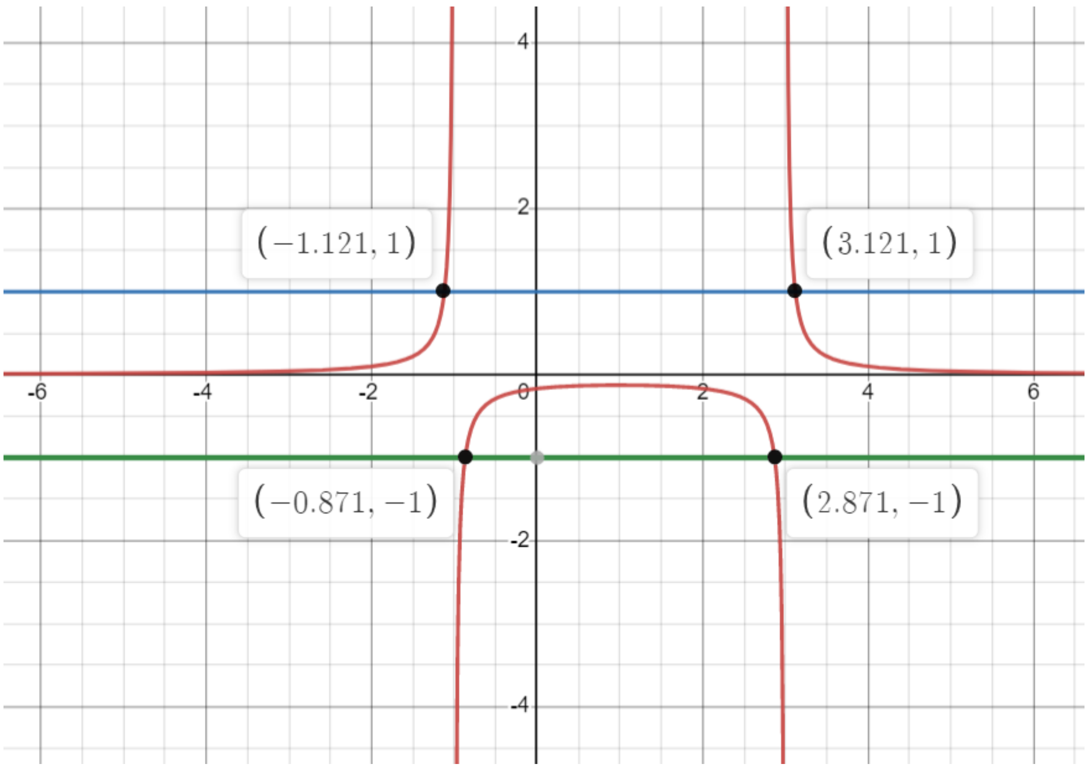
The range of y = $\frac{1}{f(x)}$ is y > 0 and

y ≤ -0.125, in both cases y ϵ R.

- **Analyze** the invariant points with respect to their quadrants. **Determine** the invariant point in quadrant 1, accurate to two decimals. [2 Marks]

Possible Solution

Invariant points occur when y = 1 and y = -1.



There are 4 invariant points, one in each quadrant.

The invariant point in quadrant 1 is (3.12, 1).