Sequence and Series Unit Exam Solutions

1. If the sum of the first 16 terms of an arithmetic sequence is 40 and the common difference is 5, then the first term in this series is

A) -9 B) -27 C) -35 D) -72

Solution

We know:

$S_{16} = 40$

$d = 5$

We are asked to find the first term, t_1. Select the appropriate arithmetic sum formula.

$S_n = \frac{n}{2}[2t_1 + (n - 1)d]$

$S_{16} = \frac{16}{2}[2(t_1) + ((16) - 1)5$

$40 = 8 [2(t_1) + 75]$

Divide both sides of the equal sign by 8.

$5 = 2(t_1) + 75$

Subtract 75 from both sides.

$-70 = 2(t_1)$

Divide both sides by 2.

$t_1 = -35$

The correct answer is C.

2. In an arithmetic sequence, the fourth term is 0 and the thirteenth term is 27. The value of d, the common difference is _3__.

Solution

The general term for an arithmetic sequence is $t_n = t_1 + (n - 1)d$.

Use the general term twice to set up two equations in two variables.

$t_4 = t_1 + ((4) - 1)d$ $t_{13} = t_1 + ((13) - 1)d$

$0 = t_1 + 3d$ $27 = t_1 + 12d$

Subtract one equation from the other to eliminate t_1.

$27 = t_1 + 12d$

$\underline{0 = t_1 + 3d}$

$27 = 9d$

$d = 3$

The value of d, the common difference is 3.

3. The first term of a geometric series is 160 and the common ratio is 1.5. If the sum of the series is 2110, then the number of terms is

A) 4 B) 5 C) 6 D) 7

Solution

The key phrase is *geometric series,* which means the corresponding sequence is geometric and the question is dealing with the sum of the terms.

We know:

$t_1 = 160$

$r = 1.5$

$S_n = 2110$.

Select the appropriate sum formula. Since we do not know the last term, t_n, we will not use $S_n = \frac{rt_n - t_1}{r-1}$, but we will use

$$S_n = \frac{t_1(r^n - 1)}{r - 1}$$

$$2110 = \frac{160((1.5)^n - 1)}{1.5 - 1}$$

$2110(0.5) = 160((1.5)^n – 1)$

$1055 = 160((1.5)^n – 1)$

$6.59375 = ((1.5)^n – 1)$

$7.59375 = (1.5)^n$

Use guess and test to find the value of n.

n = 5.

The correct answer is B.

4. In a geometric sequence, $t_1 = 17$ and $t_6 = 4131$. The common ratio for this sequence is _3_.

Solution

We can use the general term for a geometric sequence to answer this question.

$t_n = t_1(r)^{n-1}$

$t_6 = (17)(r)^{6-1}$

$4131 = 17r^5$

$243 = r^5$

Determine the 5th root of 243.

$r = \sqrt[5]{243}$

$r = 3$

The common ratio for this sequence is 3.

5. Given the geometric sequence, $-\frac{75}{32}, \frac{15}{8}, -\frac{3}{2}, \frac{6}{5}, \ldots$, the value of the common ratio can be written in the form, $-\frac{m}{k}$ where m and k are integers. The values of m and k, respectively, are

A) 5 and 9 B) 4 and 5 C) 5 and 2 D) 4 and 9

Solution

When given a geometric sequence, the common ratio can be determined by dividing any term, after the first term, by the previous term.

$$r = \frac{t_n}{t_{n-1}}$$

For example,

$$r = \frac{t_2}{t_1} = \frac{\frac{15}{8}}{-\frac{75}{32}}$$

$= \quad \left(\frac{15}{8}\right)\left(-\frac{32}{75}\right) = -\frac{480}{600}$

When simplified, the common ratio is equal to $-\frac{4}{5}$.

In the form, $-\frac{m}{k}$, m = 4 and k = 5.

The correct answer is B.

Use the following information to answer the next question.

The following statements are made regarding the infinite geometric series, $3 + 3\left(\frac{5}{4}\right) + 3\left(\frac{5}{4}\right)^2 + 3\left(\frac{5}{4}\right)^3 + \ldots$

Statement 1	The sum is 12.
Statement 2	$-1 < r < 1$
Statement 3	$t_1 = \frac{5}{4}$.
Statement 4	This series is divergent.

6. The correct statement is

A) 1 B) 2 C) 3 D) 4

Solution

Statement 1

The common ratio is $\frac{5}{4}$. Since $r > 1$, this geometric series is divergent and thus has no sum. Thus, statement 1 is **false.**

Statement 2

From statement 1, we determined that $r > 1$. Therefore, $-1 < r < 1$, is not correct. Statement 2 is **false.**

Statement 3

The first term, or t_1 is equal to 3, not $\frac{5}{4}$. Statement 3 is **false.**

Statement 4

Since $r > 1$, this series is divergent. Statement 4 is **true.**

The correct answer is D.

7. The number of terms in the arithmetic sequence, 6, $\frac{19}{3}$, $\frac{20}{3}$, …, 40, is

A) 102 B) 103 C) 104 D) 105

Solution

We know the first term, the last term, and we can calculate the common difference, d,. We will use the general term of an arithmetic sequence.

$t_n = t_1 + (n – 1)d$.

We know:

$t_1 = 6$

The common difference, d, is any term subtract the previous term. For example,

$d = \frac{19}{3} – 6$

$d = \frac{19}{3} - \frac{18}{3} = \frac{1}{3}$

$t_n = 40$

$t_n = t_1 + (n – 1)d$.

$40 = 6 + (n – 1)\frac{1}{3}$

Multiply each term by 3 to clear the fraction.

120 = 18 + (n – 1)

102 = n – 1

103 = n

The correct answer is B.

8. The sum of an infinite geometric series is $\frac{80}{3}$. If r = $\frac{1}{4}$, then the value of t_1 is __20__.

Solution

Use the formula, $S_\infty = \frac{t_1}{1-r}$, r ≠ 1.

$$\frac{80}{3} = \frac{t_1}{1-\left(\frac{1}{4}\right)}$$

$$\frac{80}{3} = \frac{t_1}{\frac{3}{4}}$$

$$t_1 = \left(\frac{80}{3}\right)\left(\frac{3}{4}\right)$$

t_1 = 20.

The value of t_1 is 20.

9. The first 3 terms of a geometric sequence are x, x + 7, 4x, … Which statement below is correct?
   A) The common ratio is 2 and the 4^{th} term is 56.
   B) The common ratio is 2 and the 4^{th} term is 28.
   C) The common ratio is 3 and the 4^{th} term is 56.
   D) The common ratio is 3 and the 4^{th} term is 28.

Solution

The common ratio is found by dividing any term, after the first term, by the term before it.

Therefore, $\frac{t_2}{t_1} = \frac{t_3}{t_2}$

Substitute the expressions for the terms. In other words,

$t_1 = x$

$t_2 = x + 7$

$t_3 = 4x$

$$\frac{x+7}{x} = \frac{4x}{x+7}$$

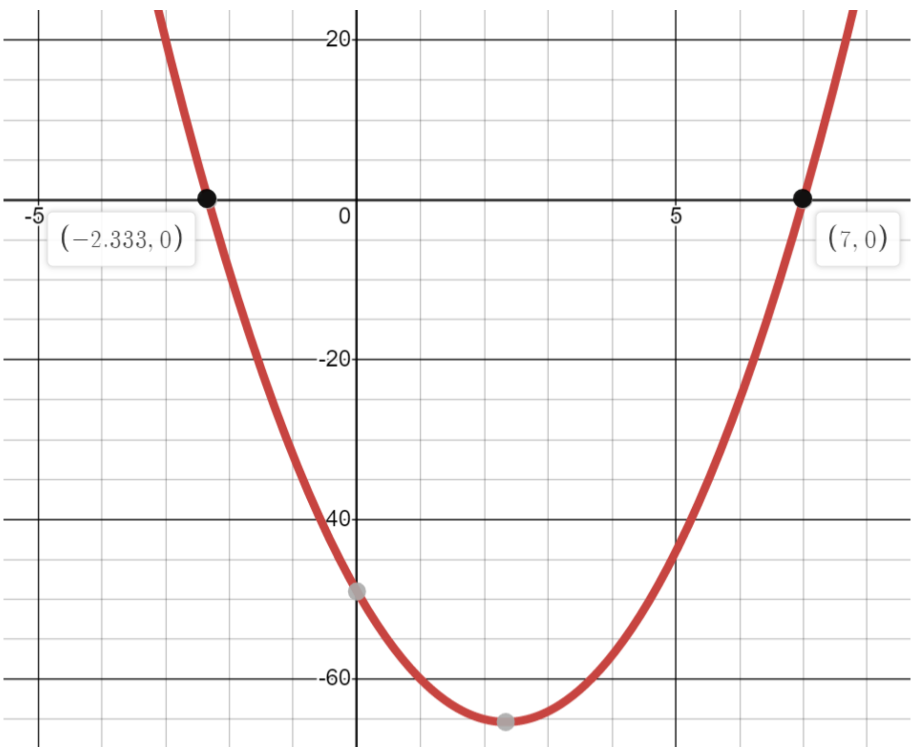
Cross multiply

$(x + 7)(x + 7) = (x)(4x)$

$x^2 + 14x + 49 = 4x^2$

$0 = 3x^2 - 14x - 49$

Graph this quadratic equation and find the x-intercepts.



In this context, the integer x = 7, makes sense.

The first four terms are, 7, 14, 28, 56.

The common ratio is 2 and the 4th term is 56.

The correct answer is A.

10. The sum of the sequence, -9, -1, 7, …, 135 can be written in the form ABCD, where each of the 4 letters represents an integer. The values of A, B, C, and D, respectively, are _1_, _1_, _9_, and _7_.

Solution

The common difference, d, is found by taking any term, other than t_1, and subtracting the previous term from it. For example,

$t_2 – t_1 = d$

$(-1) – (-9) = d$

$-1 + 9 = d$

$8 = d$

In order to utilize a sum formula, the number of terms must be known. We can use the general term of an arithmetic sequence, given that $t_1 = -9$, $d = 8$ and $t_n = 135$.

$t_n = t_1 + (n – 1)d$

$135 = -9 + (n – 1)8$

$144 = 8n – 8$

$152 = 8n$

$n = 19$

There are 19 terms in this sequence.

Since we know the first term (-9), the last term (135), and the number of terms (19) we can use the formula

$$S_n = \frac{n}{2}(t_1 + t_n)$$

$$S_{19} = \frac{19}{2}(-9 + 135)$$

$$S_{19} = \frac{19}{2}(126)$$

$S_{19} = 1197$

The values of A, B, C, and D, respectively, are 1, 1, 9, and 7.

11. If the sum of 6 terms of a geometric series is 189 and the common ratio is 2, then t_8 is

A) 192 B) 384 C) 768 D) 1536

Solution

To find the eighth term, t_8, we need to use the general term for a geometric sequence,

$t_n = (t_1)(r)^{n-1}$

We can use a sum formula to determine t_1.

Do we use

$S_n = \frac{rt_n - t_1}{r-1}$?

Or, do we use

$S_n = \frac{t_1(r^n-1)}{r-1}$?

Since we do not know the last term, t_n, we will use the second formula.

$$S_n = \frac{t_1(r^n - 1)}{r - 1}$$

$$189 = \frac{t_1(2^6 - 1)}{2 - 1}$$

$189 = t_1 (63)$

$3 = t_1$

Now use the general term for the geometric sequence.

$t_8 = (3)(2)^{8-1}$

$t_8 = (3)\ (2^7)$

$t_8 = 384.$

The correct answer is B.

12. The 5th term of a geometric sequence is 72 and the 9th term is 93 312. The 2nd term of this sequence is

$A)\ \frac{1}{18}$ B) $\frac{1}{6}$ C) $\frac{1}{3}$ D) 2

Solution

We do not know the first term or the common ratio. Set up two equations in two variables.

$t_5 = (t_1)\ (r)^{5-1}$ $t_9 = (t_1)\ (r)^{9-1}$

$72 = (t_1)\ (r)^4$ $93\ 312 = (t_1)\ (r)^8$

Divide these two equations.

$93\ 312 = (t_1)\ (r)^8$

$\underline{72 = (t_1)\ (r)^4}$

$1296 = r^4$

Take the 4th root of 1296.

$r = 6$

Substitute $r = 6$ into either of the two equations to find t_1.

$72 = (t_1)\ (6)^4$

$$\frac{72}{6^4} = t_1$$

$$\frac{1}{18} = t_1$$

$t_2 = (t_1)\ (r)^{2-1}$

$t_2 = \left(\frac{1}{18}\right)(6)$

$t_2 = \frac{1}{3}$

The second term is $\frac{1}{3}$.

The correct answer is C.

13. Tony begins a savings plan by saving $1 during the first week. In each subsequent week, he saves $3 more than the week before. At the end of the 20th week, the total amount he has saved is
    A) $58 B) $580 C) $590 D) $1160

Solution

With $1 the first week, $4 the second week, $7 the third week, an arithmetic sequence is formed. Since we are looking to sum the total amount, an arithmetic series is formed.

1 + 4 + 7 + …

The common difference, d, is 4 – 1, or 3.

The first term, t_1, is 1.

The total number of terms, n, is 20.

Use the formula, $S_n = \frac{n}{2}[2t_1 + (n-1)d]$

$S_{20} = \frac{20}{2}[2(1) + ((20) - 1)3]$

$S_{20} = 10\,[2 + (19)3]$

$S_{20} = 10\,[59]$

$S_{20} = 590$

At the end of the 20th week, the total amount saved is $590.

The correct answer is C.

14. Suppose your aunt and uncle receive yearly payments from an annuity. On each yearly anniversary, they receive 90% of the preceding years amount. If the first payment is $6350, how much in all (to the nearest dollar) will be paid out in 8 years?

A) 32 540 B) 36 165 C) 38 890 D) 42 397

Solution

A geometric series is formed:

$6350 + 6350(0.9) + 6350(0.9)^2 + \dots$

Determine S_8.

$t_1 = 6350$

$r = 0.9$

$n = 8$

$$S_n = \frac{t_1(r^n - 1)}{r - 1}$$

$$S_8 = \frac{6350((0.9)^8 - 1)}{-0.1}$$

$S_8 = \$36\ 165.33$

The correct answer is B.

Written Response

- Write your responses as neatly as possible.
- For full marks, your responses must address **all** aspects of the question.
- All responses, including descriptions and/or explanations of concepts must include pertinent ideas, calculations, formulas and correct units.
- Your responses must be presented in a in a well-organized manner. For example, you may organize your responses in point form or paragraphs.

WRITTEN RESPONSE 1

Tom was asked to find the general term for the geometric sequence, $8r^3$,$16r^4$,$32r^5$, ...

His work is shown below.

Step 1	$\frac{16r^4}{8r^3} = 2r$
Step 2	$t_n = 8r^3(2r)^{n-1}$
Step 3	$t_n = (2r)^3(2r)^{n-1}$
Step 4	$t_n = (2r)^{3n-3}$

- **Analyze** his work and **describe** the error made by Tom. [2 Marks]

Focus on the directing words and recall their meaning.

Analyze – "*Make a mathematical examination of parts to determine the nature, proportion, function, interrelationships, and characteristics of the whole".*

Describe – "*Give a written account of a concept".*

Solution

In step 1, Tom began by determining the common ratio, r. He knew that the common ratio can be determined by dividing any term, other than the first term, by the previous term. He decided to use $r = \frac{t_2}{t_1} = \frac{16r^4}{8r^3} = 2r$. This is correct.

In step 2, he utilized the general form of a geometric sequence, $t_n = (t_1)(r)^{n-1}$, by substituting the expressions for t_1 and r. This step is also correct.

In step 3, he rewrote $8r^3$ in an equivalent form of $(2r)^3$. He did this because he can now combine 2 powers into a single power. This step is correct.

His error occurred in step 4.

- Make the correction. **Explain.** [1 Mark]

Focus on the directing word and recall its meaning.

Explain – "*Make clear what is not immediately obvious or entirely known; give the cause of or reason for; make known in detail".*

Solution

Although Tom knew that two powers can be combined into one, he didn't completely remember how to do this. He did keep the base (2r) the same, but he multiplied the exponents. He should have added the exponents.

Step 4 should be:

$t_n = (2r)^{n+2}$

- **Determine** the coefficient of t_8. [1 Mark]

Focus on the directing word and recall its meaning.

Determine – "*Find a solution, to a specified degree of accuracy to a problem by showing appropriate formulas, procedures and/or calculations".*

Solution

We are asked to find a specific term, the eighth term (t_8); and then state its coefficient.

Use the general form of a geometric sequence. $t_n = (t_1)(r)^{n-1}$

We know that $t_1 = 8r^3$ and that $r = 2r$.

$t_8 = (8r^3)(2r)^{n-1}$

From bullet two above, we will use the simplified version of the general term.

$t_n = (2r)^{n+2}$

$t_8 = (2r)^{8+2}$

$t_8 = (2r)^{10}$

$t_8 = 2^{10}r^{10}$

$t_8 = 1024r^{10}$

The coefficient of t_8 is 1024.

<u>WRITTEN RESPONSE 2</u>

Domestic bees make their honeycomb by starting with a single hexagonal cell, then forming ring after ring of hexagonal cells around the initial cell. The number of cells in successive rings from an arithmetic sequence.

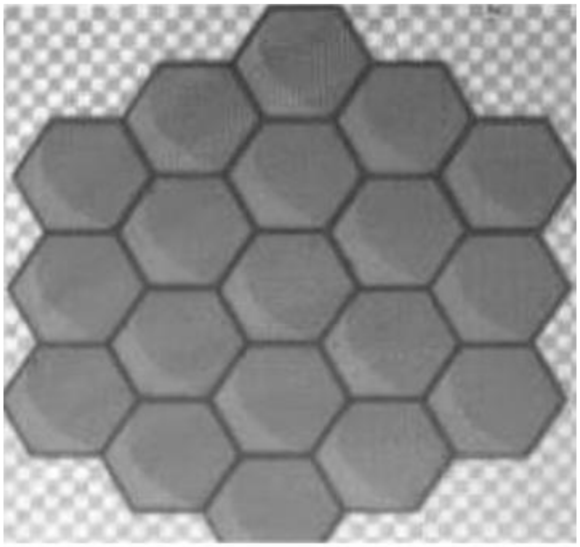
- Write a rule for the number of cells in the nth ring. **Justify.** [2 Marks]

Focus on the directing word and recall its meaning.

Justify – "*Indicate why a conclusion has been stated, by providing supporting reasons and/or evidence that form a mathematical argument".*

Solution

A hexagon has 6 sides. When a new ring is added, there are 6 more cells compared to the previous ring. The common difference, d, is 6. The first term, t_1, is 6.



Ring 1 has 6 cells.

Ring 2 has 12 cells.

Ring 3 has 18 cells.

The sequence is: 6, 12, 18, …

The general term for an arithmetic sequence is $t_n = t_1 + (n – 1)d$, where t_1 is the first term, n is the number of terms, and d is the common difference. Substitute the known values into the general term.

$t_n = 6 + (n – 1)6$

$t_n = 6 + 6n – 6$

$t_n = 6n$

NOTE – The term number corresponds to the ring number.

- **Algebraically determine** the total number of cells in the honeycomb after the 11th ring has formed. (do not forget to count the initial cell) [1 Mark]

Focus on the directing word and recall its meaning.

Algebraically – "*Using mathematical procedures that involve variables or symbols to represent values".*

Solution

When determining the total number of cells in the honeycomb, we need to use a sum formula.

However, we first need to find the number of cells in the 11th ring.

Using the general term for the arithmetic sequence, $t_n = 6n$,

$t_{11} = 6(11)$

$t_{11} = 66$

Since we know t_1 (6), the last term t_n (66), and the number of terms, n, (11), use the formula

$$S_n = \frac{n}{2}(t_1 + t_n)$$

$S_{11} = \frac{11}{2}(6 + 66)$

$S_{11} = 396$

Remember to include the first cell.

The total number of cells after the 11th ring is 397.

- If the total number of cells is 816, **determine** the number of rings that have been formed. **Sketch** a graph and **explain** how it pertains to the answer. [2 Marks]

Solution

We can't use the same formula from the bullet above,

$$S_n = \frac{n}{2}(t_1 + t_n)$$

because we do not know the last term, t_n

Instead, we will use the formula, $S_n = \frac{n}{2}[2t_1 + (n - 1)d]$

$S_n = 816$

$t_1 = 6$

$d = 6$

$816 = \frac{n}{2}[2(6) + (n - 1)6]$

Multiply both sides of the equation to clear the fraction.

$1632 = n(12 + 6n – 6)$

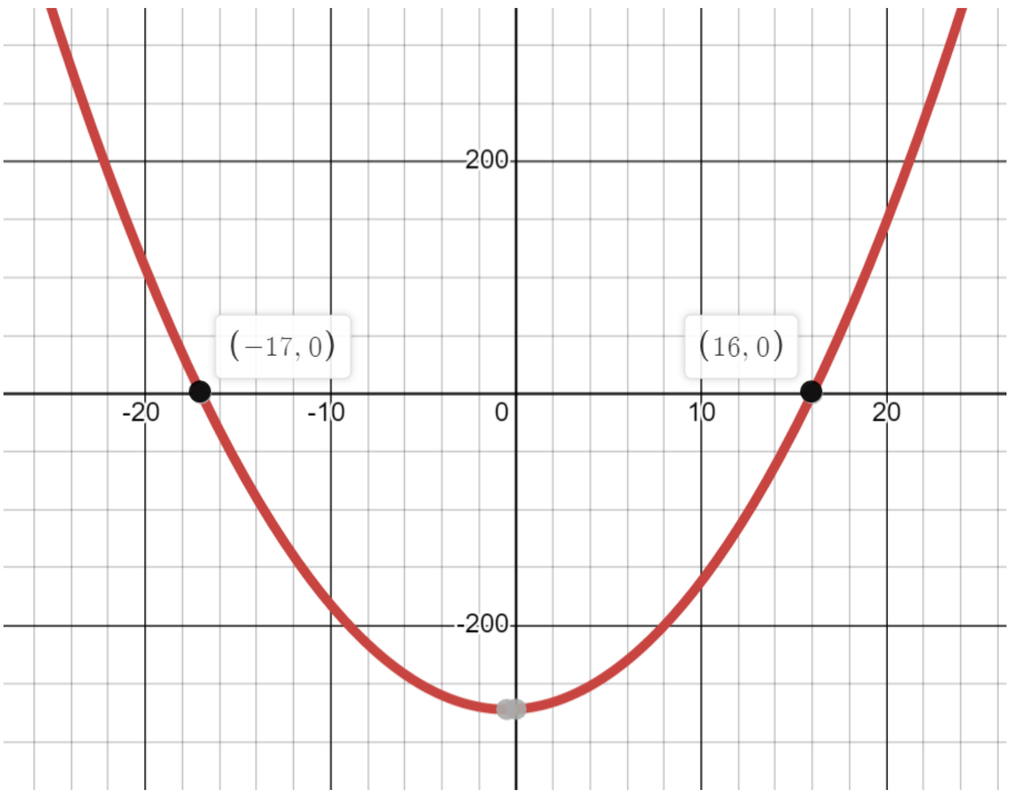
$1632 = n(6n + 6)$

$1632 = 6n^2 + 6n$

$0 = 6n^2 + 6n – 1632$

$0 = 6(n^2 + n – 272)$

Graph the equation $y = x^2 + x – 272$.



The solutions to this quadratic equation are determined by the x-intercepts. There are two x-intercepts, -17 and 16. Since the negative value doesn't make sense in this context, we will reject it. Our answer is 16.

When the total number of cells is 816, the number of rings that have been formed is 16.