Math 30-1 Exponents and Logarithms Written Response Section

- Write your responses as neatly as possible.
- For full marks, your responses must address **all** aspects of the question.
- All responses, including descriptions and/or explanations of concepts must include pertinent ideas, calculations, formulas and correct units.
- Your responses must be presented in a in a well-organized manner. For example, you may organize your responses in point form or paragraphs.

Use the following information to answer the next question.

The population growth or decay of two particular Western Canadian towns can be modelled by the functions:

$$P(t) = 5500(1.016)^t$$

$$P(t) = 7200(0.975)^t,$$

where P(t) is the population t years from 2009.

Written Response 1

- **Compare** the two functions with respect to the y-intercept. **Describe** the meaning of the y-intercept in this context. [2 Marks]

[*Compare: "Examine the character or qualities of two things by providing characteristics of both that point out their mutual similarities and differences".*]

[*Describe: "Give a written account of a concept".*]

- **Compare** the two functions with respect to their growth or decay. **Interpret** the specific growth or decay rate for each function. [3 Marks]

[*Interpret: "Provide a meaning of something; present information in a new form that adds meaning to the original data".*]

- For the growth function, **algebraically determine**, to the nearest year, when the population will reach 10 000. **Explain** your process. [3 Marks]

[*Algebraically: "Using mathematical procedures that involve variables or symbols to represent values".*]

[*Determine: "Find a solution, to a specified degree of accuracy, to a problem by showing appropriate formulas, procedures, and/or calculations".*]

[*Explain:" Make clear what is not immediately obvious or entirely known; give the cause of or reason for; make known in detail".*]

- Using technology, **determine** when the population of the two towns will be equal, to the nearest tenth of a year. **Explain** your process. Provide a final statement that **interprets** the meaning of the coordinates. [3 Marks]

Use the following information to answer the next question.

A Math 30-1 student was asked to horizontally stretch $y = \log_2 x$ by a factor of $\frac{1}{24}$ about the y-axis and then state an equivalent transformation.

Step 1 $y = \log_2 24x$

Step 2 $y = \log_2 8 + \log_2 16x$

Step 3 $y = 3 + \log_2 16x$

Step 4 Compared to $y = \log_2 x$, $y = \log_2 24x$ has been been horizontally stretched by a factor of $\frac{1}{16}$ about the y-axis and translated 3 units up.

Written Response 2

- **Analyze** the student's work and **explain** the error made by the Math 30-1 student. **Determine** the correct transformation. [3 Marks]

[*Analyze: "Make a mathematical examination of parts to determine the nature, proportion, function, interrelationships, and characteristics of the whole".*]

- How does this question-and-answer change if the original horizontal stretch was to be by a factor of 24, instead of $\frac{1}{24}$? **Justify**. [2 Marks]

[*Justify: "Indicate why a conclusion has been stated, by providing supporting reasons and/or evidence that form a mathematical argument".*]

Written Response 3

- If $m^2 = 10$, **solve** the equation for c, given, $\log_c(m+1) + \log_c(m-1) = \frac{2}{3}$. [3 Marks]

[*Solve: "Give a solution to a problem".*]

- **Verify** the solution. [2 Marks]

[*Verify: "Establish, by substitution for a particular case or by geometric comparison, the truth of a statement".*]

- If $m^2 = 1$, will there be a solution? **Explain.** [2 Marks]

Exponents and Logarithms Written Response Possible Solutions

Use the following information to answer the next question.

The population growth or decay of two particular Western Canadian towns can be modeled by the functions:

$$P(t) = 5500(1.016)^t$$

$$P(t) = 7200(0.975)^t,$$

where P(t) is the population t years from 2009.

Written Response 1

- **Compare** the two functions with respect to the y-intercept. **Describe** the meaning of the y-intercept in this context. [2 Marks]

Possible Solution

The function, $P(t) = 5500(1.016)^t$ has the smaller y-intercept of 5500. The function $P(t) = 7200(0.975)^t$ has the larger y-intercept of 7200.

The y-intercepts represent the populations of the respective towns in 2009.

- **Compare** the two functions with respect to their growth or decay. **Interpret** the specific growth or decay rate for each function. [3 Marks]

Possible Solution

In the form, $y = a(b)^x$, if $b > 1$, the function will increase in value over time.

The function, $P(t) = 5500(1.016)^t$ is the growth function, since b is 1.016. The growth rate is 1.6%.

In the form, $y = a(b)^x$, if $0 < b < 1$, the function will decrease in value over time.

The function $P(t) = 7200(0.975)^t$ is the decay function, since b is 0.975. The decay rate is (100% - 97.5%) or 2.5%.

- For the growth function, **algebraically determine**, to the nearest year, when the population will reach 10 000. **Explain** your process. [3 Marks]

Possible Solution

Substitute known values into the exponential function. Since we are solving for an exponent, we will use logarithms.

$10\,000 = 5500(1.016)^t$

$\frac{10000}{5500} = (1.016)^t$

Take the log of both sides, and move the exponent 't' to the front of the log by using the Power Law of logarithms.

$$\log\left(\frac{10000}{5500}\right) = t\log(1.016)$$

Isolate the variable.

$$\frac{\log\left(\frac{10000}{5500}\right)}{\log(1.016)} = t$$

$t = 37.66$

In 38 years from 2009, or in the year 2047, the population of the growth function will reach 10 000.

- Using technology, **determine** when the population of the two towns will be equal, to the nearest tenth of a year. **Explain** your process. Provide a final statement that **interprets** the meaning of the coordinates. [3 Marks]

Possible Solution

Graph both functions and determine the point of intersection. The window settings will likely need to be adjusted in order to see the location where the graphs will cross. Since the y-intercepts are 5500 and 7200, a possible window setting could be: [-10,10,1, -10, 8000, 800].

The intersection point is (6.5, 6101.5). This means that in about 6 and a half years, the population of both towns will be about 6100.

Use the following information to answer the next question.

A Math 30-1 student was asked to horizontally stretch $y = \log_2 x$ by a factor of $\frac{1}{24}$ about the y-axis and then state an equivalent transformation. Analyze the following steps.

Step 1 $y = \log_2 24x$

Step 2 $y = \log_2 8 + \log_2 16x$

Step 3 $y = 3 + \log_2 16x$

Step 4 Compared to $y = \log_2 x$, $y = \log_2 24x$ has been horizontally stretched by a factor of $\frac{1}{16}$ about the y-axis and translated 3 units up.

Written Response 2

- **Analyze** the student's work and **explain** the error made by the Math 30-1 student. **Determine** the correct transformation. [3 Marks]

Possible Solution

In step two, the product law was applied incorrectly. It is not two numbers that add to 24, but it is two numbers that multiply to 24.

Step 1 $y = \log_2 24x$

Step 2 $y = \log_2 8 + \log_2 3x$

Step 3 $y = 3 + \log_2 3x$

Step 4 Compared to $y = \log_2 x$, $y = \log_2 24x$ has been horizontally stretched by a factor of $\frac{1}{3}$ about the y-axis and translated 3 units up.

- How does this question and answer change if the original horizontal stretch was to be by a factor of 24, instead of $\frac{1}{24}$? **Justify**. [2 Marks]

Possible Solution

The stretch factor and the number in front of x are reciprocals of each other.

Step 1 $y = \log_2\left(\frac{1}{24}x\right)$

Using the product law, we can split the single logarithm into two terms.

Step 2 $y = \log_2\left(\frac{1}{8}\right) + \log_2\left(\frac{1}{3}x\right)$

We can replace $\log_2\left(\frac{1}{8}\right)$ with a specific value, i.e. -3.

Step 3 $y = -3 + \log_2\left(\frac{1}{3}x\right)$

Step 4 Compared to $y = \log_2 x$, $y = \log_2\left(\frac{1}{24}x\right)$ has been horizontally stretched by a factor of 3 about the y-axis and translated 3 units down.

The changes are:

- **the stretch factor is 3 instead of** $\frac{1}{3}$.
- **the translation is down 3 units instead of up 3 units.**

Written Response 3

- If $m^2 = 10$, **solve** the equation for c, given, $\log_c(m+1) + \log_c(m-1) = \frac{2}{3}$. [3 Marks]

Possible Solution

In the logarithmic equation, combine the two terms on the left into one term using the product law of logarithms.

$\log_c(m + 1)(m - 1) = \frac{2}{3}$

$\log_c(m^2 - 1) = \frac{2}{3}$

Substitute 10 for m^2.

$\log_c(10 - 1) = \frac{2}{3}$

$c^{\frac{2}{3}} = 9$

Raise each base to an exponent of $\frac{3}{2}$

$c^{\left(\frac{2}{3}\right)\left(\frac{3}{2}\right)} = 9^{\left(\frac{3}{2}\right)}$

$c = \left(\sqrt{9}\right)^3$

c = 27

- **Verify** the solution. [2 Marks]

Possible Solution

Solutions should be verified from the original equation. Since $m^2 = 10$, $m = \sqrt{10}$.

$$\log_{27}(\sqrt{10}+1) + \log_{27}(\sqrt{10}-1) = \frac{2}{3}$$

Using the change of base to get approximate solutions:

$$\frac{\log(\sqrt{10}+1)}{\log 27} + \frac{\log(\sqrt{10}-1)}{\log 27} = \frac{2}{3}$$

0.4326859905 + 0.2339806762 = 0.666…

0.666… = 0.666…

- If $m^2 = 1$, will there be a solution? **Explain.** [2 Marks]

If $m^2 = 1$, we would have $\log_c(1 - 1) = 2$

Converting to exponential form, $c^2 = 0$. In this case, $c = 0$.

However, a logarithmic function is defined only for values of c that are greater than zero, and not equal to 1.

Hence, there is no solution.